3/10/20

Module 3: Aggregate (Compound) Distributions

M3S1: IDEA

X_i = rvr the amount of the i^{th} loss (severity)

N = rvr the number of losses per exposure unit

Assume $\{X_1, X_2, \ldots, N\}$ are all mutually independent

The total (aggregate) <u>amount</u> of losses per exposure unit can be written as

$$S = X_1 + X_2 + \cdots + X_N \qquad X_i \sim X \text{ (common distribution)}$$

$$S = \sum_{i=1}^{N} X_i \qquad \left(\text{Exam P: } S = \sum_{i=1}^{n} X_i \quad n\text{-fixed (not random)}\right)$$

<u>Note</u>: $S \approx N(\mu, \sigma^2)$ (Normal Distribution)

We want to know $\mu = E[S]$ & $\sigma^2 = Var(S)$

$E[S]$: $E[S|N] = N \cdot E[X] \Longrightarrow E[S] = E[N \cdot \underbrace{E[X]}_{\#}]$

$$\boxed{\therefore E[S] = E[N] \cdot E[X]}$$

$Var(S)$: $E[S|N] = N \cdot E[X]$ and $Var(S|N) = N \cdot Var(X)$

$$\Longrightarrow Var(S) = Var(\cancelto{N \cdot E[X]}{E[S|N]}) + E[\cancelto{N \cdot Var(X)}{Var(S|N)}]$$

$$\boxed{\therefore Var(S) = (E[X])^2 \cdot Var(N) + Var(X) \cdot E[N]}$$

M352: Including Deductibles on Each Policy
(possibly other policy modifications)

S = rvr aggregate payments by insurance company per exposure unit

Recall: Y^L = payment per loss r.v. (E.g. $Y^L = (X-d)_+$)
(= 0 for all losses less than d)

Y^P = payment per payment r.v. (E.g. $Y^P = X-d \mid X>d$)
(no zero's)

$$E[(Y^L)^k] = E[(Y^P)^k] \cdot Pr(X>d) \quad \text{for any } k$$

Define N^L = rvr the # of losses per exposure unit

N^P = rvr the # of payments per exposure unit

Then $S = \sum_1^{N^L} Y_i^L$ (includes the zeros) or $S = \sum_1^{N^P} Y_i^P$ (has no zeros)

Example:

X	Pr
100	.2
400	.4
500	.4

$d = 200$

Y^L	Pr
0	.2
200	.4
300	.4

Y^P	Pr
200	.5
300	.5

$d = 200$

<u>Data</u>:

X:	100	100	400	500	100	500	500	400
Y^L:	0	0	200	300	0	300	300	200
Y^P:	–	–	200	300	–	300	300	200

$$S = \sum_{1}^{N^L} Y_i^L = 0 + 0 + 200 + 300 + 0 + 300 + 300 + 200$$

or

$$S = \sum_{1}^{N^P} Y_i^P = 200 + 300 + 300 + 300 + 200$$

Reminder Let $\pi = Pr(X > d) = Pr(\text{payment is made})$

Then

N^L	N^P
$P(\lambda)$	$P(\lambda\cdot\pi)$
$B(m, q)$	$B(m, q\cdot\pi)$
$NB(r, \beta)$	$NB(r, \beta\cdot\pi)$

<u>Example</u>: $N \sim P(\lambda)$ X - anything (Compound Poisson Distribution)

Q: $E[S]$ & $Var(S)$ $S = \sum_{1}^{N} X_i$

$$E[S] = E[N]\cdot E[X] \Longrightarrow \boxed{E[S] = \lambda\cdot E[X]}$$

$$Var(S) = (E[X])^2 \cdot \overbrace{Var(N)}^{\lambda} + Var(X)\cdot \overbrace{E[N]}^{\lambda}$$
$$= \lambda\cdot\left((E[X])^2 + Var(X)\right)$$

$$\boxed{\therefore Var(S) = \lambda\cdot E[X^2]}$$

Now Suppose $X \sim Exp(\theta = 1000)$

& $d = 100$ on each policy & $\lambda = 200 \; (= \lambda^L)$

Q: $E[S]$ & $Var(S)$

$S = \sum_1^{N^L} Y_i^L$ or $\underbrace{S = \sum_1^{N^P} Y_i^P}$

easier since if $X \sim Exp(\theta)$

method 1: then $(Y^P = X - d \mid X > d) \sim Exp(\theta)$

Method 1: $S = \sum_1^{N^P} Y_i^P$

$Y^P \sim Exp(\theta = 1000)$ & $N^P \sim P(\overbrace{\lambda \cdot \pi}^{\lambda^P})$

where $\lambda = 200$

$\pi = Pr(X > 100) = e^{-\frac{100}{1000}} = e^{-.1}$

$E[S] = \lambda^P \cdot E[Y^P] = \underline{\underline{200 \cdot e^{-.1} \cdot 1000}} = 200000 e^{-.1}$

$Var(S) = \lambda^P \cdot E[(Y^P)^2] = \underline{\underline{200 \, e^{-.1} \cdot (2 \cdot 1000^2)}}$

Method 2: $S = \sum_1^{N^L} Y_i^L$

$Y^L = (X - d)_+$ $N^L \sim P(\lambda^L = 200)$

$E[S] = \lambda^L \cdot E[Y^L] = 200 \cdot E[Y^P] \cdot Pr(X > 100) = \underline{\underline{200 \cdot 1000 \cdot e^{-.1}}}$

$Var(S) = \lambda^L \cdot E[(Y^L)^2] = 200 \cdot E[(Y^P)^2] \cdot Pr(X > 100)$

$= \underline{\underline{200 \cdot (2 \cdot 1000^2) \cdot e^{-.1}}}$

$\underline{Q}$: $Pr(S < 200000)$

$$S = N\left(\mu = 200000e^{-.1},\ \sigma^2 = 200e^{-.1} \cdot (2(1000^2))\right)$$

SND - Standard Normal Distribution

$\therefore Pr(S < 200000)$

$$= Pr\Big(SND < \underbrace{\frac{200000 - 200000e^{-.1}}{\sqrt{200e^{-.1}(2 \cdot 1000^2)}}}_{1.0004\ldots}\Big) = 0.84$$